

Darboux transformations and hidden quadratic supersymmetry of the one-dimensional stationary Dirac equation

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Abstract

A matricial Darboux operator intertwining two one-dimensional stationary Dirac Hamiltonians is constructed. This operator is such that the potential of the second Dirac Hamiltonian as well as the corresponding eigenfunctions are determined through the knowledge of only two eigenfunctions of the first Dirac Hamiltonian. Moreover this operator together with its adjoint and the two Hamiltonians generate a quadratic deformation of the superalgebra subtending the usual supersymmetric quantum mechanics. Our developments are illustrated on the free particle case and the generalized Coulomb interaction. In the latter case, a relativistic counterpart of shape-invariance is observed.

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1 Introduction

In quantum mechanics, the Schrödinger equations which can be solved by analytic methods exclusively are rather exceptional. Therefore the methods being able to enlarge the number of such equations have attracted much attention in recent as well as less recent literature. Three of them still remain very popular: the Darboux transformations [1] elaborated in 1882 within the mathematical framework of Sturm-Liouville differential equations, the factorization method introduced by Schrödinger [2] in 1940 and more recently the so-called supersymmetric quantum mechanics [3]. All of them are more or less based on the same following ideas.

Let us consider the following Schrödinger Hamiltonian

$$H_0 \equiv -\frac{d^2}{dx^2} + V_0(x), \quad x \in \mathbb{R} \text{ or } x \in \mathbb{R}_0^+ \quad (1)$$

which can be factorized as follows

$$H_0 = L^\dagger L + \alpha, \quad \alpha = \text{constant} \quad (2)$$

with

$$L = \frac{d}{dx} + W(x). \quad (3)$$

Then the eigenfunctions of the isospectral (up to the eventual creation or loss of one energy) Hamiltonian H_1 defined by

$$H_1 \equiv LL^\dagger + \alpha = -\frac{d^2}{dx^2} + V_1(x) \quad (4)$$

are obtained through the application of L to the eigenfunctions of H_0 as it is clear from the so-called intertwining relation

$$LH_0 = H_1L. \quad (5)$$

Thus a new Schrödinger Hamiltonian H_1 has been constructed and it is exactly solvable if H_0 is.

We remark that the Darboux transformation has two particular features compared to [2, 3]: First, the potential $W(x)$, written as

$$W(x) = -\frac{d \ln \psi_0(x)}{dx}, \quad (6)$$

can be constructed from a bounded *or unbounded* non-vanishing eigenfunction $\psi_0(x)$ of H_0 (with eigenvalue α). Second, the Darboux operator L can be extended to higher order [1, 4].

Here we shall ask for the same kind of developments in the relativistic context that is to say search for the operator L intertwining two one-dimensional Dirac Hamiltonians. A partial answer has already been given in

[5] and [6] through the supersymmetrical features of specific Dirac Hamiltonians. Another one can also be found in [7, 8] where a relativistic Darboux transformation has been considered but for pseudoscalar potentials only. In the following we will not limit ourselves to such a context and will give, in Section 2, the extended intertwining operator L corresponding to a general self-adjoint potential. This operator is constructed from two (known) solutions of the initial Dirac equation and gives rise to new exactly solvable Dirac equations. Moreover, in Section 3, we will convince ourselves from this operator L and its adjoint that the underlying superstructure in the relativistic context is a quadratic *deformation* of the $sqm(2)$ superalgebra, the latter being, as well known [3], the subtending superalgebra of the (non-relativistic) supersymmetric quantum mechanics. Finally, in Section 4, we will illustrate our statements on two examples: The free particle case and the generalized Coulomb interaction. For the latter, we observe the relativistic counterpart of the so-called shape-invariance [9], i.e., only the values of the parameters introduced in the expression of the potentials change.

2 Intertwining operator for the Dirac equation

Let us start with the following one-dimensional Dirac Hamiltonian

$$h_0 \equiv i\sigma_2 \frac{d}{dx} + v_0(x), \quad x \in \mathbb{R} \text{ or } x \in \mathbb{R}_0^+ \quad (7)$$

where σ_2 is the usual two-by-two Pauli matrix and v_0 is real and symmetric, i.e.,

$$v_0(x) = \begin{pmatrix} v_{11}^0(x) & v_{12}^0(x) \\ v_{12}^0(x) & v_{22}^0(x) \end{pmatrix}. \quad (8)$$

We assume here that h_0 is a known exactly solvable Hamiltonian; in other words, all its eigenfunctions, the two-component spinors $\psi(x)$, as well as the corresponding energies are analytically determined. Let us now search for a matricial operator L satisfying the intertwining relation similar to (5), i.e.,

$$Lh_0 = h_1L \quad (9)$$

with

$$h_1 \equiv i\sigma_2 \frac{d}{dx} + v_1(x), \quad (10)$$

$v_1(x)$ being at this level the unknown real and symmetric potential. The simplest operator L we can consider is

$$L \equiv A \frac{d}{dx} + B \quad (11)$$

where A and B are two-by-two matrices with x -dependent entries. The relations (9) and (11) give the following system

$$[A, \sigma_2] = 0, \quad (12)$$

$$[B, \sigma_2] - iAv_0 + iv_1A - \sigma_2A_x = 0, \quad (13)$$

$$Av_{0x} + Bv_0 - v_1B - i\sigma_2B_x = 0, \quad (14)$$

the notation A_x meaning here

$$\frac{dA}{dx} \equiv \begin{pmatrix} \frac{dA_{11}}{dx} & \frac{dA_{12}}{dx} \\ \frac{dA_{21}}{dx} & \frac{dA_{22}}{dx} \end{pmatrix}.$$

The condition (12) is equivalent to ask for $A_{11} = A_{22}$ and $A_{12} = -A_{21}$. The constraint (13) enables us to fix the potential difference $\Delta v \equiv v_1 - v_0$

$$\Delta v = (Av_0 - v_0A + i[B, \sigma_2] - i\sigma_2A_x)A^{-1} \quad (15)$$

up to the assumption of the existence of A^{-1} . Finally from (14) we can obtain the matrix B or in a simpler way σ defined through $B \equiv A\sigma$. Indeed, Eq. (14) then reads

$$(v_0 - i\sigma_2\sigma)_x + [\sigma, v_0] + i[\sigma_2, \sigma]\sigma = 0. \quad (16)$$

We recognize a matrix analogue of the Riccati equation. It can be linearized through the substitution

$$\sigma = -u_x u^{-1} \quad (17)$$

in order to become

$$[u^{-1}(v_0u + i\sigma_2u_x)]_x = 0 \quad (18)$$

which after integration leads to

$$h_0u = i\sigma_2u_x + v_0u = u\lambda, \quad (19)$$

the matrix λ being the constant of integration.

This equation (19) is thus formally speaking an ordinary Dirac one up to the fact that the solution u is not a spinor anymore but a two-by-two matrix while the usual energy E has also been replaced by a two-by-two matrix λ .

The next step is to find a convenient u that is a solution of (19) being real (and invertible) in order to ensure the self-adjointness of v_1 through (15). It is ensured in a straightforward manner if

$$u = (u_1, u_2), \quad \lambda = \text{diag}(\varepsilon_1, \varepsilon_2) \quad (20)$$

with the spinors u_1 and u_2 being eigenfunctions (not necessarily bounded) of the Dirac Hamiltonian h_0

$$h_0 u_j = \varepsilon_j u_j, \quad j = 1, 2. \quad (21)$$

Having found u , the operator L given in Eq. (11) or

$$L = A \left(\frac{d}{dx} - u_x u^{-1} \right) \quad (22)$$

as well as the new potential v_1 (see Eq. (15))

$$v_1 = A \left(v_0 + i[\sigma, \sigma_2] - i\sigma_2 A^{-1} A_x \right) A^{-1} \quad (23)$$

are now fixed up to the determination of A . This matrix keeps arbitrariness: all one knows is that it has to commute with σ_2 . For simplicity and comparison with the non-relativistic context, we put A equal to the identity matrix. Eqs. (22) and (23) are then simplified as follows

$$L = \frac{d}{dx} - u_x u^{-1}, \quad (24)$$

$$v_1 = v_0 + i[\sigma, \sigma_2]. \quad (25)$$

These results are the relativistic analogues of the usual Darboux transformation. We now give another expression in what concerns v_1 , particularly useful for applications. Indeed from (19) we have

$$\sigma = -u_x u^{-1} = i\sigma_2 u \lambda u^{-1} - i\sigma_2 v_0 \quad (26)$$

and therefore

$$v_1 = \sigma_2 v_0 \sigma_2 + u \lambda u^{-1} - \sigma_2 u \lambda u^{-1} \sigma_2, \quad (27)$$

i.e.,

$$v_1 = \sigma_2 v_0 \sigma_2 + \frac{\varepsilon_1 - \varepsilon_2}{\det u} \begin{pmatrix} d_1 & d_2 \\ d_2 & -d_1 \end{pmatrix}, \quad (28)$$

where $d_1 \equiv u_{11}u_{22} + u_{12}u_{21}$, $d_2 = u_{21}u_{22} - u_{11}u_{12}$, with u_{ij} corresponding to the element of the matrix u being at the crossing of the i^{th} line and the j^{th} column.

Let us close this Section by noticing that, by definition, the operator L has a non-trivial kernel since $\ker L = u$. This implies that the action of L to an eigenspinor of h_0 corresponding to an eigenvalue different from ε_1 and ε_2 will give rise to an eigenspinor of h_1 . The eigenspinors of h_1 related to the eigenvalues ε_1 and ε_2 will be obtained through $v \equiv (u^\dagger)^{-1}$, that is $h_1 v = v \lambda$.

3 Factorization properties of Dirac Hamiltonians and second order supersymmetry

Let us here consider in addition to L given in Eq. (24), its adjoint L^\dagger defined by

$$L^\dagger = -\frac{d}{dx} - (u_x u^{-1})^\dagger. \quad (29)$$

It satisfies an intertwining relation similar to Eq. (9)

$$L^\dagger h_1 = h_0 L^\dagger. \quad (30)$$

This relation means that the operator L^\dagger realizes the transformation in the opposite direction, i.e., the application of L^\dagger to the eigenspinors of h_1 gives us the eigenspinors of h_0 . The operator $L^\dagger L$ is thus such that applied to the eigenspinors of h_0 , it gives back these eigenspinors. By definition, this is nothing but the fact that $L^\dagger L$ is a symmetry operator of the initial Dirac equation $h_0 \psi = E \psi$. Since we limit ourselves to the one-dimensional stationary context, this implies that $L^\dagger L$ is a function of h_0 . Moreover because $L^\dagger L$ is a second order differential (matricial) operator while h_0 is of the first order, $L^\dagger L$ is in fact a polynomial of second order in h_0 . More precisely, after tedious calculations, one can be convinced that

$$L^\dagger L = (h_0 - \varepsilon_1)(h_0 - \varepsilon_2) \quad (31)$$

while a similar result holds for LL^\dagger

$$LL^\dagger = (h_1 - \varepsilon_1)(h_1 - \varepsilon_2). \quad (32)$$

If we now introduce the 4 by 4 matrices

$$H \equiv \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}, Q^\dagger = \begin{pmatrix} 0 & L^\dagger \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix} \quad (33)$$

the relations (9) and (30)–(32) can be reformulated as

$$[Q, H] = [Q^\dagger, H] = 0, \{Q, Q^\dagger\} \equiv QQ^\dagger + Q^\dagger Q = (H - \varepsilon_1)(H - \varepsilon_2) \quad (34)$$

while

$$Q^2 = (Q^\dagger)^2 = 0. \quad (35)$$

Relations (34)–(35) are the ones of a quadratic deformation of the superalgebra $sqm(2)$ subtending the usual supersymmetric quantum mechanics [3]. This quadratic superalgebra cannot be seen directly from the Dirac equation and therefore we associate it with a hidden supersymmetry. Let us also and finally notice that a superalgebra similar to the one of (34)–(35) can also be found in the non-relativistic context when considering second order Darboux transformations [10].

4 Examples

Let us now turn to some examples and see how our method provides us new exactly solvable Dirac potentials from known ones.

4.1 The free particle case

We consider here the potential

$$v_0(x) = m\sigma_1, \quad x \in \mathbb{R}. \quad (36)$$

Note that it corresponds to an unusual -but convenient- realization (the usual one being associated with $v_0(x) = m\sigma_3$) of the Clifford algebra subtending the one-dimensional Dirac equation. As stated in (20), it is necessary to take account of two eigenspinors corresponding to (36). Let u_1 and u_2 defined by

$$u_1 = \begin{pmatrix} \text{ch}(kx) + \frac{cE}{k}\text{sh}(kx) \\ \text{ch}(kx + 2\alpha) + \frac{cE}{k}\text{sh}(kx + 2\alpha) \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\text{ch}(kx) \\ \text{ch}(kx + 2\alpha) \end{pmatrix}, \quad (37)$$

be such eigenfunctions (of respective eigenvalues $\varepsilon_1 = E$ and $\varepsilon_2 = -E$) with

$$k = \sqrt{m^2 - E^2}, \quad e^{2\alpha} = \sqrt{\frac{m - k}{m + k}}, \quad c = \text{constant}. \quad (38)$$

The unique constraint to take care of in order to apply our method is to have a nonvanishing determinant: $\det u \neq 0$. Here it is precisely given by

$$\det u = \frac{1}{E} \left[m + E \text{ch}(2kx + 2\alpha) + \frac{E^2 c}{k} \text{sh}(2kx + 2\alpha) \right] \equiv \frac{1}{E} \Delta \quad (39)$$

and the parameter c is such that $|c| < k/E$ in order to satisfy this constraint. The result (28) then gives rise to the new exactly solvable potential v_1

$$v_1(x) = \frac{2E^2 c}{\Delta} \sigma_3 + \left(m - \frac{2k^2}{\Delta} \right) \sigma_1 \quad (40)$$

whose eigenspinors can be obtained from the application of L defined in Eq. (24) to the eigenspinors of the free Dirac Hamiltonian. Notice that the potential $v_1(x)$ given in Eq. (40) reduces to the well known one-soliton scalar potential when $c = 0$.

4.2 The generalized Coulomb case

Before going to this example, we would like to mention that the usual radial equation associated to the (3+1)-dimensional Dirac equation is included in our developments. Indeed, the standard radial equation when coupled to scalar $W(x)$ and vector $V(x)$ potentials is

$$\left\{ \frac{d}{dx} - \frac{k}{x} \sigma_3 + [M + W(x)] \sigma_1 + i[E - V(x)] \sigma_2 \right\} \psi(x) = 0, \quad x \in \mathbb{R}_0^+ \quad (41)$$

where M and E are the mass and the energy of the particle while k is related to the total angular momentum. Eq. (41) can also be written as

$$\left\{ i\sigma_2 \frac{d}{dx} + \frac{k}{x} \sigma_1 + [M + W(x)] \sigma_3 - [E - V(x)] \right\} \psi(x) = 0, \quad (42)$$

which coincides with $h_0\psi(x) = E\psi(x)$ with h_0 defined in Eq. (7) and

$$v_{12}^0(x) = \frac{k}{x}, \quad v_{11}^0(x) = M + V(x) + W(x), \quad v_{22}^0(x) = -M + V(x) - W(x). \quad (43)$$

Let us now turn to our example. It corresponds to the choices of Ref. [11]:

$$V(x) = \frac{\alpha}{x}, \quad W(x) = \frac{\beta}{x}. \quad (44)$$

We refer to this example as the generalized Coulomb one because the choice $(\alpha = \frac{1}{137}, \beta = 0)$ leads to the standard Coulomb interaction.

Let $\psi(x) = (\psi_1(x), \psi_2(x))^T$ be a solution of Eq. (41) or equivalently (42), when the interactions (44) are taken into account. Using standard developments, we easily find the solutions in terms of hypergeometric confluent functions

$$\begin{aligned} \psi_1(x) = & e^{-\lambda_n x} x^\mu \left[-n {}_1F_1(1-n, 2\mu+1; 2\lambda_n x) \right. \\ & \left. - \left(-k + \frac{\alpha M}{\lambda_n} + \frac{\beta E_n}{\lambda_n} \right) {}_1F_1(-n, 2\mu+1; 2\lambda_n x) \right], \quad (45) \end{aligned}$$

$$\begin{aligned} \psi_2(x) = & -\frac{\lambda_n}{M + E_n} e^{-\lambda_n x} x^\mu \left[-n {}_1F_1(1-n, 2\mu+1; 2\lambda_n x) \right. \\ & \left. + \left(-k + \frac{\alpha M}{\lambda_n} + \frac{\beta E_n}{\lambda_n} \right) {}_1F_1(-n, 2\mu+1; 2\lambda_n x) \right], \quad (46) \end{aligned}$$

where the parameters λ_n and μ are constrained by

$$\lambda_n^2 = M^2 - E_n^2, \quad (47)$$

$$\mu^2 = k^2 + \beta^2 - \alpha^2 \quad (48)$$

while the number n is defined by

$$n = - \left(\frac{\alpha E_n}{\lambda_n} + \frac{\beta M}{\lambda_n} + \mu \right). \quad (49)$$

This relation can be solved for the energies E_n as

$$E_n = \frac{-\alpha\beta \pm (n + \mu)\sqrt{\alpha^2 + (n + \mu)^2 - \beta^2}}{[\alpha^2 + (n + \mu)^2]} M, \quad (50)$$

the plus or minus sign, as well as the values taken by n , having possibly to be chosen in order to ensure the square-integrability of $\psi_1(x)$ and $\psi_2(x)$.

The most straightforward way to apply our method is to choose

$$u_1 = \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} \right) \Big|_{n=0}, u_2 = \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} \right) \Big|_{n=1}. \quad (51)$$

In order to avoid heavy notations, we rewrite these choices as

$$u_1 = \left(\begin{array}{c} x^\mu e^{-\lambda_0 x} \\ c_1 x^\mu e^{-\lambda_0 x} \end{array} \right), u_2 = \left(\begin{array}{c} x^\mu e^{-\lambda_1 x} (1 - c_2 x) \\ c_1 x^\mu e^{-\lambda_1 x} (1 - c_3 x) \end{array} \right) \quad (52)$$

with λ_0 and λ_1 defined through Eqs. (49) and (50), while

$$c_1 = \frac{\mu - k}{\alpha - \beta}, \quad (53)$$

$$c_2 = \frac{\lambda_1}{1 + 2\mu} + \frac{(E_1 + M)(\mu - k)}{(\alpha - \beta)(1 + 2\mu)}, \quad (54)$$

$$c_3 = \frac{\lambda_1}{1 + 2\mu} + \frac{(M - E_1)(\alpha - \beta)}{(\mu - k)(1 + 2\mu)}. \quad (55)$$

Applying finally the result (28), we obtain a (new) exactly solvable potential of the type

$$\begin{aligned} v_1(x) &= \frac{\alpha}{x} + \left[-M + \frac{(\varepsilon_1 - \varepsilon_2)}{c_2 - c_3} \left(\frac{2}{x} - c_2 - c_3 \right) \right] \sigma_3 \\ &+ \left\{ -\frac{k}{x} + \frac{(\varepsilon_1 - \varepsilon_2)}{c_2 - c_3} \left[\left(c_1 - \frac{1}{c_1} \right) \frac{1}{x} + \left(\frac{c_2}{c_1} - c_1 c_3 \right) \right] \right\} \sigma_1. \end{aligned} \quad (56)$$

In other words, we obtain a shape-invariant potential with respect to $v_0(x)$.

A particular example corresponding to the choices

$$\alpha = 1, \beta = -1, \mu = 1, k = 1 \quad (57)$$

can be useful to illustrate the results here. Indeed we have

$$\lambda_0 = 0, \lambda_1 = \frac{4}{5}M, c_1 = 0, c_2 = \frac{4}{15}M, c_1 c_3 = \frac{8}{15}M. \quad (58)$$

The corresponding energies are

$$\varepsilon_1 \equiv E_0 = M, \quad \varepsilon_2 \equiv E_1 = -\frac{3}{5}M. \quad (59)$$

The resulting potential is given by Eq. (56), i.e.,

$$v_1(x) = \frac{1}{x} + \left(\frac{3M}{5} + \frac{1}{x} \right) \sigma_3 + \left(\frac{2}{x} - \frac{4M}{5} \right) \sigma_1. \quad (60)$$

One can then determine the operator L , as defined in Eq. (24), connecting the eigenfunctions related to $v_0(x) = (1/x) + (M - 1/x)\sigma_3 + (1/x)\sigma_1$ and to $v_1(x)$ given in Eq. (60), respectively. It is given by

$$L = \begin{pmatrix} \frac{d}{dx} - \frac{1}{x} & \frac{2M}{5} - \frac{2}{x} \\ 0 & \frac{d}{dx} + \frac{4M}{5} - \frac{2}{x} \end{pmatrix}. \quad (61)$$

Due to its definition, it is clear that $Lu_1 \equiv Lu_2 = 0$, with u_1 and u_2 of Eq. (52) with the values (58). The definition of L also implies that, whenever applied to any of the functions $\psi(x)$, it will give the eigenfunctions corresponding to $v_1(x)$ as expressed in Eq. (60). For instance, for $n = 2$, we have

$$\begin{aligned} & L \left[\left(-\frac{2}{25} \exp^{-\frac{3M}{5}x} \right) \begin{pmatrix} 50x - 30Mx^2 + 3M^2x^3 \\ 3(-10Mx^2 + 3M^2x^3) \end{pmatrix} \right] \\ &= -\frac{6}{125} \exp^{-\frac{3M}{5}x} M^2x^2 \begin{pmatrix} -10 + 3Mx \\ 5 + 3Mx \end{pmatrix}, \end{aligned} \quad (62)$$

and one can directly check that this is a solution of the final equation $h_1\psi(x) = E\psi(x)$ with $E = -\frac{4}{5}M$. The other values of n ($= 3, 4, \dots$) evidently lead to similar results.

The last information we mention here is the possibility of obtaining new exactly solvable potentials and not only shape-invariant ones. This situation arises for example when we choose

$$u_1 = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \Big|_{n=1}, \quad u_2 = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \Big|_{n=2}. \quad (63)$$

With the set of parameters (57), we obtain

$$\begin{aligned} v_1(x) &= \frac{1}{50x - 15Mx^2 + 12M^2x^3} \\ &\begin{pmatrix} 100 + 90Mx - 60M^2x^2 & 100 - 115Mx - 27M^2x^2 + 12M^3x^3 \\ 100 - 115Mx - 27M^2x^2 + 12M^3x^3 & -120Mx + 84M^2x^2 \end{pmatrix} \end{aligned} \quad (64)$$

which has a different shape with respect to $v_0(x)$ and to the $v_1(x)$ given in Eq. (60). Evidently, one can determine the eigenfunctions related to this potential $v_1(x)$ defined in Eq. (64) through the application of the corresponding Darboux operator L on the eigenfunctions $\psi(x)$ of h_0 . One can also proceed in a similar way with different values of n and obtain families of new exactly solvable potentials $v_1(x)$ whose eigenfunctions will be known through the application of the ad-hoc Darboux operator on the solutions of the generalized Coulomb problem.

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